

# PERVERSE COHERENT SHEAVES (AFTER DELIGNE)

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**ABSTRACT.** This note is mostly an exposition of an unpublished result of Deligne [D], which introduces an analogue of perverse  $t$ -structure [BBD] on the derived category of coherent sheaves on a Noetherian scheme with a dualizing complex. Construction extends to the category of coherent sheaves equivariant under an action of an algebraic group; though proof of the general statement in this case does not require new ideas, it provides examples (such as sheaves on the nilpotent cone of a semi-simple group equivariant under the adjoint action) where construction of coherent “intersection cohomology” sheaves works.

## 1. INTRODUCTION

Let  $X$  be a reasonable stratified topological space; or let  $X$  be a reasonable scheme, stratified by locally closed subschemes. Let  $D$  be the full subcategory in, respectively, derived category of sheaves on  $X$ , or in the derived category of étale sheaves on  $X$ , consisting of complexes smooth along the stratification.

For an integer-valued function  $p$  (perversity) on the set of strata Beilinson, Bernstein and Deligne [BBD] defined a  $t$ -structure on the category  $D$ ; the objects of corresponding abelian category (core of the  $t$ -structure) are called perverse sheaves.

The question addressed in this note is whether an analogous construction can be carried out for the derived category of coherent sheaves on a reasonable scheme. Surprisingly, the answer is positive (with some modifications), easy, and not widely known (though was known to Deligne for a long time, see [D]).

Let us summarize the difference between the coherent case considered here, and the constructible case treated in [BBD].

First, in the coherent case we can not work with complexes “smooth” along a given stratification, for the corresponding subcategory in  $D^b(Coh)$  is not a full triangulated subcategory. (If  $f$  is a function whose divisor intersects the open stratum, then the cone of the morphism  $\mathcal{O} \xrightarrow{f} \mathcal{O}$  has singularity on the open stratum). This forces us to define perversity as a function on the set of generic points of all irreducible subvarieties, i.e. on the topological space of a scheme.

The second, more essential difference is that in the derived category of coherent sheaves the functor  $j^*$  of pull-back under an open imbedding  $j$  does not have adjoint functors. Recall that in constructible situation the right adjoint to  $j^*$  is the functor  $j_*$  of direct image, and the left adjoint is the functor  $j_!$  of extension by zero. In coherent set-up the functor  $j_*$  is defined in the larger category of quasi-coherent sheaves (Ind-coherent sheaves), while  $j_!$  is defined in the Grothendieck dual category introduced in Deligne’s appendix to [H] (consisting of Pro-coherent sheaves).

It turns out, however, that in the proof of the existence of perverse  $t$ -structure one can use instead of the object  $j_!(\mathcal{F})$  (where  $j : U \hookrightarrow X$  is an open imbedding)

any extension  $\tilde{\mathcal{F}}$  of  $\mathcal{F}$  to  $X$ , such that the restriction of  $\tilde{\mathcal{F}}$  to  $X - U$  has no cohomology above certain degree (depending on the perversity function). If the perversity function is *monotone* (see Definition 3 below) it is very easy to construct such  $\tilde{\mathcal{F}}$ . Applying the Grothendieck-Serre duality to this construction, we get a substitute for  $j_*(\mathcal{F})$ , which exists if the perversity function is comonotone. Otherwise the proof is parallel to that in [BBD].

Thus the  $t$ -structure is constructed not for an arbitrary perversity function, but only for a monotone and comonotone one. (In the topological situation one also needs this condition to get a  $t$ -structure on the whole derived category of constructible sheaves, rather than on the category corresponding to a fixed stratification.)

In [D] Deligne used the Grothendieck's Finiteness Theorem ([SGA2], VIII.2.1) to show that the formulas for  $\tau_{\leq 0}^p, \tau_{\geq 0}^p$  of [BBD], a priori making sense in a larger category containing  $D^b(Coh)$ , give in fact objects of  $D^b(Coh)$ , provided the perversity function is monotone and comonotone (see also Remark 3).

The results on the existence of a “perverse”  $t$ -structure carry over to the case of  $G$ -equivariant coherent sheaves, where  $G$  is a (reasonable) algebraic group acting on a (reasonable) scheme. In this case perversity  $p(x)$  must be assigned only to points  $x$  of the scheme, which are invariant under the connected component of identity of  $G$ , as an equivariant sheaf is anyway “smooth along the orbits.”

Although the general formalism for the equivariant category is very similar to the non-equivariant one (to the extent that we found it easier not to treat the two cases separately), there is one construction which works in the equivariant case only. Namely, the definition of the minimal (Goresky-MacPherson, or IC) extension functor  $j_{!*}$  works only when the perversity function is *strictly* monotone and comonotone. Though formally the proof of this statement works both in the equivariant and non-equivariant (=equivariant with trivial  $G$ ) situations, the statement can be nonempty in the equivariant case only. Indeed, it is easy to see, that a strictly monotone and comonotone perversity function exists only if  $G$  acts on the scheme with finite number of orbits, and dimensions of two adjacent orbits differ at least by two. If this is the case, an obvious analogue of the usual description of irreducible perverse sheaves as minimal extensions of local systems is valid, and the core of the  $t$ -structure is Artinian (in contrast with the core of the standard  $t$ -structure). An example of this situation is provided by the nilpotent cone of a semi-simple algebraic group over a field of characteristic zero, equipped with the adjoint action (see Remarks at the end of the note).

The exposition would probably look better (and work in greater generality) if the notion of a stack was used; however, my ignorance confined me to the language of equivariant sheaves (rather than the equivalent language of sheaves on the quotient stack).

It should be quite clear from the above that this paper does not contain original results of the author.

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## 2. PRELIMINARIES

In this section we collect some standard Lemmas needed in the exposition. The reader familiar with basic algebraic geometry certainly will not need our proofs.

Let  $X$  be a scheme over a base scheme  $S$ ; we denote the category of coherent (respectively, quasi-coherent) sheaves on  $X$  by  $Coh_X$ ,  $QuasiCoh_X$  (or simply  $Coh$ ,  $QuasiCoh$  if confusion is not likely). Let  $G$  be an affine group scheme over  $S$ , acting on  $X$ .

We will assume that  $S$  is Noetherian,  $X$ ,  $G$  are of finite type over  $S$ , and  $S$  admits a dualizing complex (in the sense of [H] §V.2); the structure morphism  $f_G : G \rightarrow S$  is assumed to be flat of finite type and Gorenstein (i.e.  $f_G^!(\mathcal{O}_S)$  is locally free).

The category of  $G$ -equivariant coherent (respectively, quasi-coherent) sheaves on  $X$  is denoted by  $Coh_X^G$ ,  $QuasiCoh_X^G$ . The forgetful functor  $Forg : QuasiCoh^G \rightarrow QuasiCoh$  has the right adjoint  $Av : \mathcal{F} \mapsto a_* pr^* \mathcal{F}$ , where  $pr : G \times X \rightarrow X$  and  $a : G \times X \rightarrow X$  are respectively the projection and the action. (Here  $Av$  stands for “averaging”.) Since  $G$  is affine and flat,  $Av$  is exact, and the canonical morphism  $\mathcal{F} \rightarrow Av(\mathcal{F})$  is an injection. Hence  $QuasiCoh^G$  has enough injectives, because  $QuasiCoh$  does.

**Lemma 1.** *Any  $G$  equivariant quasi-coherent sheaf  $\mathcal{F}$  on  $X$  is the union of its  $G$ -equivariant coherent subsheaves.*

*Proof* Let  $a : \mathcal{F} \rightarrow \mathcal{O}(G) \otimes_{\mathcal{O}(S)} \mathcal{F}$  denote the coaction. For a coherent (possibly non-equivariant) subsheaf  $\mathcal{F}_0 \subset \mathcal{F}$  let  $\mathcal{F}_0^G \subset \mathcal{F}$  be the preimage under  $a$  of  $\mathcal{O}(G) \otimes_{\mathcal{O}(S)} \mathcal{F}_0$ . Then one readily checks that  $\mathcal{F}_0^G$  is an equivariant coherent subsheaf, and that  $\bigcup_{\mathcal{F}_0} \mathcal{F}_0^G = \mathcal{F}$ .  $\square$

**Corollary 1.** *For  $? = b$ , or  $-$  the category  $D^?(Coh^G)$  is equivalent to the full subcategory of  $D^?(QuasiCoh^G)$  consisting of complexes with coherent cohomology.*

*Proof* It suffices to check that for a bounded above complex  $\mathcal{F}^\bullet$  of equivariant quasicoherent sheaves, whose cohomology is coherent, the set of quasiisomorphic equivariant coherent subcomplexes in  $\mathcal{F}^\bullet$  is nonempty and filtered under inclusion; and that any equivariant coherent subcomplex in  $\mathcal{F}^\bullet$  lies in an equivariant coherent quasiisomorphic subcomplex. This follows from Lemma 1 by a standard argument. Namely, let  $\mathcal{Z}^i, \mathcal{B}^i \subset \mathcal{F}^i$  denote, respectively, the kernel and the image of the differential. We construct by descending induction in  $i$  a coherent equivariant subsheaf  $\mathcal{F}_c^i \subset \mathcal{F}^i$  satisfying the two properties:  $d(\mathcal{F}_c^i) = \mathcal{F}_c^{i+1} \cap \mathcal{B}^{i+1}$ ; and  $(\mathcal{F}_c^i \cap \mathcal{Z}^i) \rightarrow \mathcal{Z}^i / \mathcal{B}^i = \mathcal{H}^i$ . If we are given a coherent subcomplex  $\mathcal{G}^\bullet \subset \mathcal{F}^\bullet$  we can choose  $\mathcal{F}_c^i$  to satisfy also  $\mathcal{F}_c^i \supset \mathcal{G}^i$ .  $\square$

We will denote the full subcategories in  $D^+(QuasiCoh)$ ,  $D^+(QuasiCoh^G)$  consisting of complexes with coherent cohomology by  $D_c^+(QuasiCoh)$ ,  $D_c^+(QuasiCoh^G)$  respectively.

**Corollary 2.** *Let  $U \subset X$  be an open  $G$ -invariant subscheme. For any  $\mathcal{F}, \mathcal{G} \in D^b(Coh^G(U))$  and a morphism  $f : \mathcal{F} \rightarrow \mathcal{G}$  there exist  $\tilde{\mathcal{F}}, \tilde{\mathcal{G}} \in D^b(Coh^G(X))$ , and a*

morphism  $\tilde{f} : \tilde{\mathcal{F}} \rightarrow \tilde{\mathcal{G}}$ , such that  $\tilde{f}|_U \cong f$ . For any two  $\tilde{\mathcal{F}}', \tilde{\mathcal{F}}'' \in D^b(\text{Coh}^G(X))$  and an isomorphism  $f : \tilde{\mathcal{F}}'|_U \cong \tilde{\mathcal{F}}''|_U$  there exists  $\tilde{\mathcal{F}} \in D^b(\text{Coh}^G(X))$ , and morphisms  $f' : \tilde{\mathcal{F}}' \rightarrow \tilde{\mathcal{F}}$ ,  $f'' : \tilde{\mathcal{F}}'' \rightarrow \tilde{\mathcal{F}}$  such that  $f'|_U, f''|_U$  are isomorphisms, and  $f''|_U \circ f' = f|_U$ .

*Proof* Let  $\mathcal{F}^\bullet$  be a finite complex of equivariant quasicoherent sheaves on  $X$ , such that cohomology of  $\mathcal{C}^\bullet|_U$  is coherent. A construction similar to the one used in the proof of Corollary 1 shows that the set of coherent equivariant subcomplexes  $\mathcal{F}_c^\bullet \subset \mathcal{F}^\bullet$ , such that imbedding of  $\mathcal{F}_c^\bullet|_U \hookrightarrow \mathcal{F}^\bullet|_U$  is quasiisomorphism, is nonempty and filtered under inclusions. Moreover, any equivariant coherent subcomplex in  $\mathcal{F}^\bullet$  lies in such a subcomplex. The statement follows.  $\square$

We will write  $X^{\text{top}}$  for the topological space of a scheme  $X$ . If  $x \in X^{\text{top}}$  is a point of  $X$  (respectively  $Z \subset X$  is a locally closed subscheme), then  $i_x : \{pt\} \hookrightarrow X^{\text{top}}$  (respectively  $i_Z : Z \hookrightarrow X$ ) will denote the imbedding.

We will use the same notation for a functor on an abelian category and its derived functor. In particular, for  $x \in X^{\text{top}}$  the functors  $i_x^* : D^?(Coh) \rightarrow D^?(O_x - mod)$ , and  $i_x^! : D^+(QuasiCoh) \rightarrow D^+(O_x - mod)$  are derived of respectively an exact, and of a left exact functor. The functor  $i_x^!$  factors through the derived category of torsion  $\mathcal{O}_x$  modules, and has finite homological dimension (because so does the functor  $j_*$  where  $j$  is an open imbedding in a Noetherian scheme).

Everywhere below we will assume that  $Coh$  has enough locally free objects. Also, dealing with equivariant categories, we will assume that  $Coh^G$  has enough locally free objects.

Then both in  $D(Coh)$  and in  $D(Coh^G)$  internal Hom (denoted by  $\underline{Hom}$ ) can be computed as derived functor in either of the two variables, and commutes with the forgetful functor from the equivariant to the nonequivariant category; also for a ( $G$ -equivariant) morphism  $f : Z \rightarrow X$  the coherent pull-back functor  $f^*$  is defined in both categories and commutes with forgetful functor.

**Lemma 2.** *Let  $Z \subset X$  be a locally closed ( $G$ -invariant) subscheme, and  $n$  be an integer. Let  $x \in X^{\text{top}}$  be a generic point of  $Z$ . Then*

a) *For  $\mathcal{F} \in D^-(Coh^G)$  we have  $i_x^*(\mathcal{F}) \in D^{\leq n}(O_x - mod)$  iff there exists an open ( $G$ -invariant) subscheme  $Z^0 \subset Z$ ,  $Z^0 \ni x$ , such that  $i_{Z^0}^*(\mathcal{F}) \in D^{\leq n}(Coh_{Z^0}^G)$ ;*

b) *For  $\mathcal{F} \in D_c^+(QuasiCoh^G)$  we have  $i_x^!(\mathcal{F}) \in D^{\geq n}(O_x - mod)$  iff there exists an open ( $G$ -invariant) subscheme  $Z^0 \subset Z$ ,  $Z^0 \ni x$ , such that  $i_{Z^0}^!(\mathcal{F}) \in D^{\geq n}(Coh_{Z^0}^G)$ .*

*Proof* Existence of an open ( $G$ -invariant) subscheme  $Z^0 \subset Z$  as in (a) is equivalent to  $i_x^* i_Z^*(\mathcal{F}) \in D^{\leq n}((O_Z)_x - mod)$ . (Indeed, if the last equality holds, then we can let  $Z^0$  be the complement in  $Z$  to support of  $\mathcal{H}^k(i_Z^*(\mathcal{F}))$ ,  $k > n$ ; the converse is obvious.)

We can rewrite  $i_x^* i_Z^*(\mathcal{F}) = i_x^*(\mathcal{F}) \otimes_{\mathcal{O}_x}^L \mathcal{O}(Z)_x$ . Since the functor of tensor product with  $\mathcal{O}(Z)_x$  over  $\mathcal{O}_x$  is right exact, and kills no finitely generated  $\mathcal{O}_x$  modules by the Nakayama Lemma, we see that the top cohomology of  $i_x^*(\mathcal{F}) \otimes_{\mathcal{O}_x}^L \mathcal{O}(Z)_x$  and of  $i_x^*(\mathcal{F})$  occur in the same degree. This proves (a).

Similarly, the second condition in (b) says that  $i_x^! i_Z^!(\mathcal{F}) = i_x^* i_Z^!(\mathcal{F}) \in D^{\geq n}((O_Z)_x - mod)$  (the equality here is, of course, due to the fact  $x$  is generic in  $Z$ ). We rewrite  $i_x^! i_Z^!(\mathcal{F}) = RHom_{\mathcal{O}_x}(\mathcal{O}(Z)_x, i_x^!(\mathcal{F}))$ , and see that the lowest cohomology of  $i_x^!(\mathcal{F})$  and of  $RHom_{\mathcal{O}_x}(\mathcal{O}(Z)_x, i_x^!(\mathcal{F}))$  occur at the same degree, because

$\mathrm{Hom}_{\mathcal{O}_x}(\mathcal{O}(Z)_x, \_)$  is left exact, and kills no torsion module, while cohomology of  $i_x^!(\mathcal{F})$  is a torsion  $\mathcal{O}_x$ -module.  $\square$

**Lemma 3.** *Let  $i : \mathbf{Z} \hookrightarrow X^{\mathrm{top}}$  be imbedding of a closed  $G$ -invariant subspace.*

a) (cf. e.g. [H], Theorem V.4.1) *For any  $\mathcal{F} \in D^-(\mathrm{Coh}^G)$ ,  $\mathcal{G} \in D^+(\mathrm{QuasiCoh}^G)$  we have*

$$\mathrm{Hom}(\mathcal{F}, i_* i^!(\mathcal{G})) = \varinjlim_Z \mathrm{Hom}(\mathcal{F}, i_{Z*} i_Z^!(\mathcal{G})),$$

where  $Z$  runs over the set of closed  $G$ -invariant subschemes of  $X$  with the underlying topological space  $\mathbf{Z}$ .

b) *If  $\mathcal{F} \in D^b(\mathrm{Coh}^G)$  is such that the cohomology sheaves  $\mathcal{H}^i(\mathcal{F})$  are supported on  $\mathbf{Z}$ , then there exists a closed  $G$ -invariant subscheme  $Z \subset X$ ,  $Z^{\mathrm{top}} = \mathbf{Z}$ , such that  $\mathcal{F} \cong i_{Z*}(\mathcal{F}_Z)$  for some  $\mathcal{F}_Z \in \mathrm{Coh}^G(Z)$ .*

*Proof* a) Let us represent  $\mathcal{F}$  by a bounded above complex  $P_{\mathcal{F}}$  of locally free coherent equivariant sheaves, and  $\mathcal{G}$  by a bounded below complex  $I_{\mathcal{G}}$  of injective quasicoherent equivariant sheaves. If  $\mathcal{I}$  is an injective object of  $\mathrm{QuasiCoh}^G(X)$ , and  $Z \subset X$  is a closed  $G$ -invariant subscheme, then  $i_Z^!(\mathcal{I}) \in \mathrm{QuasiCoh}^G(Z)$  is injective; hence locally free equivariant sheaves on  $X$  are adjusted to  $\mathrm{Hom}(\_, i_{Z*} i_Z^!(\mathcal{I}))$ . Thus  $R\mathrm{Hom}(\mathcal{F}, i_{Z*} i_Z^!(\mathcal{G})) = R\mathrm{Hom}(i_Z^*(\mathcal{F}), i_Z^!(\mathcal{G}))$  is computed by the complex  $\mathrm{Hom}^\bullet(P_{\mathcal{F}}, i_{Z*} i_Z^!(I_{\mathcal{G}}))$ . On the other hand,  $R\mathrm{Hom}(\mathcal{F}, i_* i^!(\mathcal{G}))$  is computed by the complex  $\mathrm{Hom}^\bullet(P_{\mathcal{F}}, i_* i^!(I_{\mathcal{G}}))$ , as  $i_* i^!$  sends injective objects of  $\mathrm{QuasiCoh}^G$  into injective ones. We have  $i_* i^!(I_{\mathcal{G}}) = \bigcup_Z i_{Z*} i_Z^!(I_{\mathcal{G}})$ ; and also  $\mathrm{Hom}^\bullet(P_{\mathcal{F}}, i_* i^!(I_{\mathcal{G}})) = \bigcup_Z \mathrm{Hom}^\bullet(P_{\mathcal{F}}, i_{Z*} i_Z^!(I_{\mathcal{G}}))$ , because  $P_{\mathcal{F}}$  is a bounded above complex of coherent sheaves. This implies the Lemma.

b) The category  $\mathrm{QuasiCoh}_{\mathbf{Z}}^G(X)$  of  $G$ -equivariant quasi-coherent sheaves supported on  $Z$  has enough injectives; moreover they are also injective as objects of the larger category  $\mathrm{QuasiCoh}^G(X)$  (this follows from the corresponding statement for non-equivariant sheaves, since  $Av$  preserves sheaves supported on  $\mathbf{Z}$ ). Hence (see e.g. [H], Proposition I.4.8)  $\mathcal{F}$  is quasiisomorphic to a finite complex of quasi-coherent sheaves supported on  $\mathbf{Z}$ . As in the proof of Corollary 1 we can represent this complex as a union of quasiisomorphic equivariant coherent subcomplexes; any such subcomplex is supported on a closed subscheme  $Z$ ,  $Z^{\mathrm{top}} \subset \mathbf{Z}$ .  $\square$

**Definition 1.** An equivariant dualizing complex on  $X$  is an object  $\mathbb{D}\mathbb{C}^G \in D^b(\mathrm{Coh}^G)$ , such that every  $\mathcal{F} \in D^b(\mathrm{Coh}^G)$  is  $\mathbb{D}\mathbb{C}$ -reflexive, i.e. the natural transformation  $\mathcal{F} \rightarrow \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(\mathcal{F}, \mathbb{D}\mathbb{C}), \mathbb{D}\mathbb{C})$  is an isomorphism.

**Lemma 4.**  $\mathcal{F} \in D^b(\mathrm{Coh}^G)$  is an equivariant dualizing complex iff  $\mathrm{Forg}(\mathcal{F})$  is a dualizing complex.

*Proof* The 'if' direction is clear because  $\underline{\mathrm{Hom}}$  commutes with the forgetful functor. The 'only if' follows from [H], Proposition V.2.1, which says, in particular, that if the structure sheaf  $\mathcal{O}$  is  $\mathbb{D}\mathbb{C}$  reflexive, then  $\mathbb{D}\mathbb{C}$  is a dualizing complex. Since  $\mathcal{O}$  obviously lies in the image of the forgetful functor, we see that  $\mathrm{Forg}(\mathbb{D}\mathbb{C}^G)$  is a dualizing complex.  $\square$

**Proposition 1.** *In the above assumptions  $X$  admits an equivariant dualizing complex.*

*Proof* According to [BBD], Theorem 3.2.4 an object of the derived category of sheaves on a cite can be given locally provided negative local Ext's from the object to itself vanish. Applying it to the covering  $G \times X \rightarrow X$  in the cite of flat  $G$ -schemes

over  $X$ , we see that it is enough to provide an isomorphism  $\pi^*(\mathbb{D}\mathbb{C}) = a^*(\mathbb{D}\mathbb{C})$  (here that  $\pi : G \times X \rightarrow X$ , and  $a : G \times X \rightarrow X$  are the projection and the action maps), satisfying an associativity constraint on  $G \times G \times X$ . Since  $f_G : G \rightarrow S$  is Gorenstein, the sheaf  $f_G^!(\mathcal{O}_S)$  is invertible; the group structure on  $G$  provides then a canonical isomorphism  $f_G^* = f_G^!$  (as follows e.g. from Remark in [H], pp 143-144). Hence  $\pi^*(\mathbb{D}\mathbb{C}) = a^*(\mathbb{D}\mathbb{C})$  are both canonically isomorphic to  $f_{G \times X}^!(\mathbb{D}\mathbb{C}_S)$ , which provides the desired isomorphism. The associativity constraint follows from functorial properties of  $f^!$ .  $\square$

*Remark 1.* Suppose that we make an additional assumption that the structure morphism  $X \rightarrow B$  is *equivariantly embeddable*, i.e. can be presented as a composition  $X \xrightarrow{\iota} \tilde{X} \rightarrow B$ , where  $\tilde{X}$  is a smooth  $B$ -scheme with a  $G$ -action, and  $\iota$  is a  $G$ -equivariant closed imbedding (the Sumihiro embedding Theorem [Su] (see also [KKLV]) guarantees that this assumption is satisfied if  $S$  is the spectrum of an algebraically closed field of characteristic 0, and  $X$  is a normal quasiprojective variety). Then the Proposition becomes evident, for we can set  $\mathbb{D}\mathbb{C}_X^G \stackrel{\text{def}}{=} \iota^!(\Omega_{\tilde{X}}^{\text{top}})$ , the definition of  $\iota^!$  for a closed imbedding being straightforward.

### 3. PERVERSE COHERENT SHEAVES

**3.1. Construction of the  $t$ -structure.** We keep the assumptions of section 2. Not to repeat the same argument twice, we treat the equivariant case from the very beginning; the reader willing to restrict to the non-equivariant case should just let  $G$  be the trivial group (and skip 3.2 as containing no non-empty statement).

*We change the notations.* From now on *Coh*, *QuasiCoh* will denote the category of  $G$ -equivariant coherent (respectively, quasicohherent) *equivariant* sheaves on  $X$ . Also  $X^{\text{top}}$  will denote a subset in the topological space of the scheme  $X$ , consisting of generic points of  $G$ -invariant subschemes; we will endow  $X^{\text{top}}$  with the induced topology. Thus  $X^{\text{top}}$  maps to the topological space of  $S$ , and for  $s \in S$  the fiber over  $s$  is the set of points of  $X_s$  which are invariant under the component of identity in  $G_s$ .

We will say that  $x, y \in X^{\text{top}}$  are equivalent (and write  $x \sim y$ ) if  $x \in G(y)$  (i.e. if  $x \in Z^{\text{top}} \iff y \in Z^{\text{top}}$  for a  $G$ -invariant subscheme  $Z \subset X$ ). The set of equivalence classes  $X^{\text{top}}/\sim$  is identified with the set of points of the stack  $X/G$ .

According to Proposition 1,  $X$  has an equivariant dualizing complex; we fix one, denote it by  $\mathbb{D}\mathbb{C}$ . This choice defines the *codimension function*  $d$  on (all) points of  $X$ , which is determined by the condition that  $i_x^!(\mathbb{D}\mathbb{C})$  is concentrated in homological degree  $d$  (see [H], §V.7). We set  $\dim(x) = -d(x)$ ; if, say,  $X$  is of finite type over a field, we can (and will) assume that  $\dim(x)$  is the (Krull) dimension of the closure of  $x$ . Notice that  $\dim(x) = \dim(y)$  for  $x, y \in X^{\text{top}}$ ,  $x \sim y$ .

Let  $\tau_{\leq n}^{\text{stand}} : D^?(Coh) \rightarrow D^{\leq n}(Coh)$ ,  $\tau_{\geq n}^{\text{stand}} : D^?(Coh) \rightarrow D^{\geq n}(Coh)$  be the truncation functors with respect to the usual  $t$ -structure on  $D^?(Coh)$ . (Here  $? = +, -$  or  $b$ .)

Let  $p$  (perversity) be an integer-valued function on  $X^{\text{top}}$ , constant on equivalence classes.

We define the dual perversity by  $\bar{p}(x) = -\dim(x) - p(x)$ .

**Definition 2.** We define  $D^{p, \leq 0} \subset D^-(Coh)$ ,  $D^{p, \geq 0} \subset D_c^+(QuasiCoh)$  by:

- $\mathcal{F} \in D^{p, \geq 0}$  if for any  $x \in X^{\text{top}}$  we have  $i_x^!(\mathcal{F}) \in D^{\geq p(x)}(\mathcal{O}_x - \text{mod})$ .
- $\mathcal{F} \in D^{p, \leq 0}$  if for any  $x \in X^{\text{top}}$  we have  $i_x^*(\mathcal{F}) \in D^{\leq p(x)}(\mathcal{O}_x - \text{mod})$ .

**Lemma 5.** a)  $\mathbb{D}(D^{p,\leq 0}) = D^{\overline{p},\geq 0}$ .

b) Let  $i_Z : Z \hookrightarrow X$  be a locally closed ( $G$ -invariant) subscheme. Then  $p$  defines the induced perversity  $p_Z = p \circ i_Z : Z^{top} \rightarrow \mathbb{Z}$ . We have:

$i_Z^*$  sends  $D^{p,\leq 0}$  to  $D^{p_Z,\leq 0}$ ;  $i_Z^!$  sends  $D^{p,\geq 0}$  to  $D^{p_Z,\geq 0}$ ;

c) In the situation of (b) assume that  $Z$  is closed. Then  $i_{Z*}$  sends  $D^{p_Z,\leq 0}$  to  $D^{p,\leq 0}$ , and  $D^{p_Z,\geq 0}$  to  $D^{p,\geq 0}$ .

*Proof* a) One knows from [H], §V.6 that for any  $\mathcal{F}$  in the bounded derived category of coherent sheaves we have  $i_x^!(\mathbb{D}(\mathcal{F})) = \text{Hom}_{\mathcal{O}_x}(i_x^*(\mathcal{F}), I_{\mathcal{O}_x})[-\dim(x)]$ , where  $I_{\mathcal{O}_x}$  is the injective hull of the residue field of  $\mathcal{O}_x$ . Since  $\text{Hom}_{\mathcal{O}_x}(\_, I_{\mathcal{O}_x})$  is exact and kills no finitely generated  $\mathcal{O}_x$  module, (a) follows.

b) follows from Lemma 2; in view of this Lemma if  $\mathcal{F} \in D^{p,\leq 0}$ , then for any  $x \in Z^{top} \subset X^{top}$  there exists a subscheme  $Z' \subset Z$  with generic point  $x$ , such that  $i_{Z'}^*(\mathcal{F}) = i_{Z'}^*(i_Z^*(\mathcal{F})) \in D^{\leq p(x)}(\text{Coh}_{Z'})$ , which implies  $i_Z^*(\mathcal{F}) \in D^{p_Z,\leq 0}$ ; and similarly for  $i_Z^!(\mathcal{F})$ .

c) is obvious.  $\square$

**Proposition 2.** For  $\mathcal{F} \in D^{p,\leq 0}$ ,  $\mathcal{G} \in D^{p,>0}$  we have  $\text{Hom}(\mathcal{F}, \mathcal{G}) = 0$ .

*Proof* We proceed by Noetherian induction in  $X$ ; thus we can assume that the statement with  $(X, p)$  replaced by  $(Z, p_Z)$  for a closed ( $G$ -invariant) subscheme  $Z \subsetneq X$  is known. (Otherwise replace  $X$  by a minimal closed ( $G$ -invariant) subscheme for which it is false).

Fix  $\mathcal{F} \in D^{p,\leq 0}$ ,  $\mathcal{G} \in D^{p,>0}$ . Let  $x$  be a generic point of  $X$ . Using Lemma 2 we find an open ( $G$ -invariant) subscheme  $j : U \hookrightarrow X$  containing  $x$ , such that  $j^*(\mathcal{F}) \in D^{\leq p(x)}(\text{Coh}(U))$  and  $j^*(\mathcal{G}) = j^!(\mathcal{G}) \in D^{> p(x)}(\text{Coh}(U))$ . Thus, of course,  $\text{Hom}(j^*(\mathcal{F}), j^*(\mathcal{G})) = 0$ .

Let  $i$  denote the closed imbedding of  $X^{top} - U^{top}$  into  $X^{top}$ . Consider the distinguished triangle in  $D^b(\text{QuasiCoh})$ :  $i_* i^! \mathcal{G} \rightarrow \mathcal{G} \rightarrow j_* j^*(\mathcal{G}) \rightarrow i_* i^! \mathcal{G}[1]$ . By Lemma 3(a) we see that

$$\text{Hom}(\mathcal{F}, i_* i^! \mathcal{G}) = \varinjlim_Z \text{Hom}(\mathcal{F}, i_{Z*} i_Z^!(\mathcal{G})) = \varinjlim_Z \text{Hom}(i_Z^*(\mathcal{F}), i_Z^!(\mathcal{G})) = 0,$$

because  $i_Z^*(\mathcal{F}) \in D^{p_Z,\leq 0}$ ,  $i_Z^!(\mathcal{G}) \in D^{p_Z,>0}$  by Lemma 5b), so  $\text{Hom}(i_Z^*(\mathcal{F}), i_Z^!(\mathcal{G})) = 0$  by the induction hypotheses. This implies the desired equality  $\text{Hom}(\mathcal{F}, \mathcal{G}) = 0$ , since  $\text{Hom}(\mathcal{F}, j_* j^*(\mathcal{G})) = \text{Hom}(j^*(\mathcal{F}), j^*(\mathcal{G})) = 0$ .  $\square$

**Definition 3.** A perversity function  $p$  is

*monotone* if  $x' \in \overline{x} \Rightarrow p(x') \geq p(x)$ ;

*strictly monotone* if  $x' \in \overline{x} \Rightarrow p(x') > p(x)$ ;

(strictly) *comonotone* if the dual perversity  $\overline{p}(x) = -\dim(x) - p(x)$  is (strictly) comonotone.

**Theorem 1.** Suppose that a perversity  $p$  is monotone and comonotone. Then  $(D^{p,\leq 0} \cap D^b, D^{p,\geq 0} \cap D^b)$  define a  $t$ -structure on  $D^b(\text{Coh})$ .

*Proof* In view of Proposition 2 we have only to show, that for any  $\mathcal{F} \in D^b(\text{Coh})$  there exists a distinguished triangle  $\mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}''$  with  $\mathcal{F}' \in D^{p,\leq 0}$ ,  $\mathcal{F}'' \in D^{p,>0}$ . We again proceed by Noetherian induction; thus we can assume that for a closed ( $G$ -invariant) subscheme  $Z \subsetneq X$ , and  $\mathcal{F} \in D^b(\text{Coh}_Z)$  there exists a triangle  $\mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}''$  with  $\mathcal{F}' \in D^{p_Z,\leq 0}(\text{Coh}_Z)$ ,  $\mathcal{F}'' \in D^{p_Z,>0}(\text{Coh}_Z)$ .

It will be convenient to use the following notation (see [BBD], 1.3.9). If  $D'$ ,  $D''$  are sets of (isomorphism classes) of objects of a triangulated category  $D$ , then

$D' * D''$  is the set of (isomorphism classes) of objects of  $D$ , defined by the condition:  $B \in D' * D''$  iff there exists a distinguished triangle  $A \rightarrow B \rightarrow C$  with  $A \in D'$ ,  $C \in D''$ . The octahedron axiom implies (see [BBD], Lemma 1.3.10) that the  $*$  operation is associative, i.e.  $(D' * D'') * D''' = D' * (D'' * D''')$ . Thus the meaning of the notation  $D_1 * \dots * D_n$  is unambiguous.

We will make the following abuse of notations: for a category  $\mathcal{A}$  we will write  $\mathcal{A}$  instead of “the set of isomorphism classes of  $Ob(\mathcal{A})$ ”; and for a functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  we will write  $F(\mathcal{A})$  instead “image of the map from the set of isomorphism classes of  $Ob(\mathcal{A})$  to that of  $Ob(\mathcal{B})$  induced by  $F$ ”. Then the statement we want to prove says that

$$D = D^{p, \leq 0} * D^{p, > 0}. \quad (1)$$

We claim that it is enough to show that

$$D = \bigcup_Z D^{p, \leq 0} * i_{Z*}(D^b(Coh_Z)) * D^{p, > 0}, \quad (2)$$

where  $Z$  runs over all ( $G$ -invariant) closed subschemes  $Z \subsetneq X$ . Indeed, by the induction assumption we know that  $D^b(Coh_Z) = D^{pz, \leq 0}(Coh_Z) * D^{pz, > 0}(Coh_Z)$ . Thus (2) implies that

$$D = \bigcup_Z D^{p, \leq 0} * i_{Z*}(D^{pz, \leq 0}(Coh_Z)) * i_{Z*}(D^{pz, > 0}(Coh_Z)) * D^{p, > 0}.$$

Rewriting the latter expression as

$$\bigcup_Z (D^{p, \leq 0} * i_{Z*}(D^{pz, \leq 0}(Coh_Z))) * (i_{Z*}(D^{pz, > 0}(Coh_Z)) * D^{p, > 0}),$$

and noting that by Lemma 5(c) we have  $i_{Z*}(D^{pz, \leq 0}(Coh_Z)) \subset D^{p, \leq 0}$ , hence

$$D^{p, \leq 0} * i_{Z*}(D^{pz, \leq 0}(Coh_Z)) \subset D^{p, \leq 0} * D^{p, \leq 0} = D^{p, \leq 0},$$

and similarly for  $D^{p, > 0}$ , we get (1).

Let us prove (2). Fix  $\mathcal{F} \in D$ , and a generic point  $x$  of  $X$ . Let  $\bar{j} : \overline{G(x)} \hookrightarrow X$  be the imbedding of the closure of  $G(x)$  (in particular, if  $X$  is irreducible, then  $\bar{j} = id$ ). We set  $\mathcal{F}^- = \tau_{\leq p(x)}^{stand}(\bar{j}_! j^*(\mathcal{F}))$ . Thus  $\mathcal{F}^- \in D^{p, \leq 0}$ , because it is supported on  $\overline{G(x)}$ , and  $p$  is monotone. Also we have a canonical morphism  $\mathcal{F}^- \rightarrow \mathcal{F}$ . Let  $\mathcal{F}_1$  be its cone; then  $i_x^*(\mathcal{F}_1) \in D^{> p(x)}(\mathcal{O}_x - mod)$ .

The dual procedure (in the sense of Grothendieck-Serre duality) gives  $\mathcal{F}^+ \in D^{p, > 0}$ , and a morphism  $f : \mathcal{F}_1 \rightarrow \mathcal{F}^+$ , such that  $i_x^*(f)$  is an isomorphism. More precisely, we set

$$\mathcal{F}^+ = \mathbb{D}(\tau_{< \bar{p}(x)}^{stand} \bar{j}_! j^*(\mathbb{D}(\mathcal{F}_1))).$$

Since  $p$  is comonotone, we see by Lemma 5(a) that  $\mathcal{F}^+ \in D^{p, > 0}$ . Since the local duality for the Artinian ring  $\mathcal{O}_x$  is an exact functor ([H], §V.6), we see that  $i_x^*(\mathbb{D}(\mathcal{F}_1)) \in D^{\leq \bar{p}(x)}(\mathcal{O}_x - mod)$ , thus  $i_x^*(\mathbb{D}(\mathcal{F}^+)) \xrightarrow{\sim} i_x^*(\mathbb{D}(\mathcal{F}_1))$ , and hence also  $i_x^*(\mathcal{F}_1) \xrightarrow{\sim} i_x^*(\mathcal{F}^+)$ .

Thus, if we set  $\mathcal{F}^0 = cone(\mathcal{F}_1 \rightarrow \mathcal{F}^+)[-1]$ , then  $i_x^*(\mathcal{F}^0) = 0$ . Hence by Lemma 3(b) we have  $\mathcal{F}^0 \cong i_{Z*}(\mathcal{F}_Z)$  for some closed ( $G$ -invariant) subscheme  $Z \subsetneq X$ , and an object  $\mathcal{F}_Z \in D^b(Coh_Z)$ . So we get

$$\mathcal{F} \in \{\mathcal{F}^-\} * \{\mathcal{F}^0\} * \{\mathcal{F}^+\} \subset D^{p, \leq 0} * i_{Z*}(D^b(Coh_Z)) * D^{p, > 0},$$

which proves (2).  $\square$

*Remark 2.* Construction of an object  $\mathcal{F}^+ \in D^{p, >0}$  with given generic fiber (and with a morphism from a given object) is the only place in this paper, where the (equivariant) duality formalism is used.

**Corollary 3.** <sup>1</sup> *Let  $j : U \hookrightarrow X$  be an open subscheme,  $p : X^{top} \rightarrow \mathbb{Z}$  be a monotone and comonotone perversity, and  $\mathcal{F} \in D^{p, \geq 0}(Coh(U))$ . Consider  $j_*(\mathcal{F}) \in D^b(QuasiCoh(X))$ , and let  $n = \min_{x \notin U^{top}} p(x)$ . Then  $\tau_{\leq n-2}^{stand}(j_*(\mathcal{F}))$  has coherent cohomology.*

*Proof* Let  $\tilde{\mathcal{F}} \in D^b(Coh)$  be any extension of  $\mathcal{F}$  (see Corollary 2); replacing  $\tilde{\mathcal{F}}$  by  $\tau_{\geq 0}^p(\tilde{\mathcal{F}})$  we can achieve that  $\tilde{\mathcal{F}} \in D^{p, \geq 0}$ . If  $\mathbf{Z}$  denotes  $X^{top} - U^{top}$ , we can consider the exact triangle  $i_{\mathbf{Z}*}i_{\mathbf{Z}}^!(\tilde{\mathcal{F}}) \rightarrow \tilde{\mathcal{F}} \rightarrow j_*(\mathcal{F})$ . Since  $\tilde{\mathcal{F}}$  has coherent cohomology, it is enough to show that  $\tau_{\leq n-2}^{stand}(i_{\mathbf{Z}*}i_{\mathbf{Z}}^!(\tilde{\mathcal{F}})[1]) \in D^b(Coh)$  as well. However, the assumption  $\tilde{\mathcal{F}} \in D^{p, \geq 0}$  implies that  $i_{\mathbf{Z}}^!(\tilde{\mathcal{F}}) \in D^{p, \geq 0}(Z) \subset D^{\geq n}(Coh_Z)$  for  $Z^{top} = \mathbf{Z}$ , hence  $i_{\mathbf{Z}}^! \in D^{\geq n}(QuasiCoh)$ , and  $\tau_{\leq n-2}^{stand}(i_{\mathbf{Z}*}i_{\mathbf{Z}}^!(\tilde{\mathcal{F}})[1]) = 0$ .  $\square$

*Remark 3.* It was pointed out to us by Deligne that Corollary 3 is equivalent to the Grothendieck Finiteness Theorem, [SGA2], VIII.2.1.

**3.2. Coherent IC-sheaves.** We will assume that  $p$  is a monotone and comonotone perversity function. We will denote the core of the  $t$ -structure on  $D^b(Coh(X))$  constructed in the previous section by  $\mathcal{P} = \mathcal{P}_X = \mathcal{P}_{X,p}$ .

**Lemma 6.** *Let  $\mathbf{Z} = Z_0^{top} \subset X^{top}$  for a closed ( $G$ -invariant) subscheme  $Z_0 \subset X$ . Let  $\mathcal{F} \in \mathcal{P}_X$ .*

- a) *The following conditions are equivalent:*
  - i)  $i_x^*(\mathcal{F}) \in D^{< p(x)}(\mathcal{O}_x - mod)$  for all  $x \in \mathbf{Z}$ .
  - ii)  $i_Z^*(\mathcal{F}) \in D^{p_Z, < 0}$  for any closed ( $G$ -invariant) subscheme  $Z \subset X$ ,  $Z^{top} \subset \mathbf{Z}$ .
  - iii)  $Hom(\mathcal{F}, \mathcal{G}) = 0$  for all  $\mathcal{G} \in \mathcal{P}$ , such that  $supp(\mathcal{G}) \cap X^{top} \subset \mathbf{Z}$ ;
- b) *The following conditions are equivalent:*
  - i)  $i_x^!(\mathcal{F}) \in D^{> p(x)}(\mathcal{O}_x - mod)$  for all  $x \in \mathbf{Z}$ .
  - ii)  $i_Z^!(\mathcal{F}) \in D^{p_Z, > 0}$  for any closed ( $G$ -invariant) subscheme  $Z \subset X$ ,  $Z^{top} \subset \mathbf{Z}$ .
  - iii)  $Hom(\mathcal{G}, \mathcal{F}) = 0$  for all  $\mathcal{G} \in \mathcal{P}$ , such that  $supp(\mathcal{G}) \cap X^{top} \subset \mathbf{Z}$ .

*Proof* (a,i)  $\iff$  (a,ii), follows from Lemma 2 (a). For a closed subscheme  $Z$  and an object  $\mathcal{G} \in D^b(Coh_Z)$  we have  $Hom(\mathcal{F}, i_{Z*}(\mathcal{G})) = Hom(i_Z^*(\mathcal{F}), \mathcal{G})$ . If  $\mathcal{F} \in \mathcal{P}$ , then  $i_Z^*(\mathcal{F}) \in D^{p_Z, \leq 0}$ ; however, for an object of any triangulated category with a  $t$ -structure, and an object  $A \in D^{\leq 0}$  we have  $A \in D^{< 0} \iff Hom(A, B) = 0$  for all  $B$  in the core of the  $t$ -structure. This shows (a,ii)  $\iff$  (a,iii). Thus (a) is proved, and the proof of (b) is similar.  $\square$

It is convenient to reformulate the conditions of Lemma 6 as follows. Let the auxiliary perversity functions  $p^- = p_{(\mathbf{Z})}^-$ ,  $p^+ = p_{(\mathbf{Z})}^+$  be given by  $p^-(x) = p(x) = p^+(x)$  if  $x \notin \mathbf{Z}$ , and  $p^-(x) = p(x) - 1$ ,  $p^+(x) = p(x) + 1$  if  $x \in \mathbf{Z}$ . Then conditions (a) of Lemma 6 just say that  $\mathcal{F} \in D^{p^-, \leq 0}$ , and conditions (b) say that  $\mathcal{F} \in D^{p^+, > 0}$ .

**Theorem 2.** *Let  $j : U \hookrightarrow X$  be a ( $G$ -invariant) locally closed subscheme, set  $p^- = p_{(\overline{U}-U)}^-$ ,  $p^+ = p_{(\overline{U}-U)}^+$  and define a full subcategory  $\mathcal{P}_{!*}(U) \subset \mathcal{P}_{\overline{U}}$  by  $\mathcal{P}_{!*}(U) = D^{p^-, \leq 0}(Coh_{\overline{U}}) \cap D^{p^+, \geq 0}(Coh_{\overline{U}})$ .*

<sup>1</sup>This statement and idea of proof are copied from a message by Deligne to the author. (Possible mistakes belong to the author).

Suppose that  $p(x^-) > p(x)$ ,  $\bar{p}(x') > \bar{p}(x)$  for any  $x \in U^{top}$ ,  $x' \in \bar{x} \cap \bar{U}^{top}$ ,  $x' \notin U^{top}$ . Then  $j^*$  induces an equivalence between  $\mathcal{P}_{!*}(U)$  and  $\mathcal{P}_U$ .

The inverse equivalence is denoted by  $j_{!*} : \mathcal{P}_U \rightarrow \mathcal{P}_{!*}(U) \subset \mathcal{P}_{\bar{U}}$ , and is called the functor of minimal (or Goresky-MacPherson, or IC) extension.

*Proof* The conditions of the Theorem say that both  $p^-$  and  $p^+$  induce monotone and comonotone perversity functions on  $\bar{U}^{top}$ ; hence they define  $t$ -structures on  $D^b(Coh_{\bar{U}})$ . Let  $\tau^-$ ,  $\tau^+$  be the corresponding truncation functors.

We first introduce an auxiliary functor  $J_{!*}$  on  $D^b(Coh_{\bar{U}})$  by  $J_{!*} = \tau_{\geq 0}^- \circ \tau_{\geq 0}^+$ .

**Lemma 7.** a)  $J_{!*}$  takes values in  $\mathcal{P}_{!*}(U)$ .

b) If  $f : \mathcal{F} \rightarrow \mathcal{G}$  is a morphism in  $D^b(Coh_{\bar{U}})$ , such that  $j^*(f)$  is an isomorphism, then  $J_{!*}(f)$  is an isomorphism.

*Proof* It is obvious that  $J_{!*}(\mathcal{F}) \in D^{p^-, \leq 0}$ ; also, if  $\mathcal{F}_1$  denotes  $\tau_{\geq 0}^+(\mathcal{F}_1)$ , then we have an exact triangle  $\tau_{> 0}^-(\mathcal{F}_1)[-1] \rightarrow J_{!*}(\mathcal{F}) \rightarrow \mathcal{F}_1$ . Here certainly  $\mathcal{F}_1 \in D^{p^+, \geq 0}$ ; and also  $\tau_{> 0}^-(\mathcal{F}_1)[-1] \in D^{p^-, \geq 2} \subset D^{p^+, \geq 0}$ . This proves (a).

If  $f$  is as in (b), then also  $j^*(J_{!*}(f))$  is an isomorphism. But then  $J_{!*}(f)$  is a morphism in  $\mathcal{P}_{!*}(U)$ , such that  $j^*(f)$  is an isomorphism; thus its kernel and cokernel are objects of  $\mathcal{P}_{\bar{U}}$  supported on  $\bar{U} - U$ . However, Lemma 6 says that  $J_{!*}(\mathcal{F})$ ,  $J_{!*}(\mathcal{G})$  have no subobjects or quotients supported on  $\bar{U} - U$ .  $\square$

Now, using Corollary 2, we see from Lemma 7 that there exists a canonically defined functor  $\hat{j}_{!*} : D^b(Coh_U) \rightarrow \mathcal{P}_{!*}(U)$ , equipped with an isomorphism  $\hat{j}_{!*} \circ j^* = J_{!*}$ . We set  $j_{!*} = \hat{j}_{!*}|_{\mathcal{P}_U}$ . Then it is clear that  $j^* \circ j_{!*} = id_{\mathcal{P}_U}$  canonically. Also  $j_{!*} \circ j^*|_{\mathcal{P}_{!*}(U)} = id$  canonically, because  $J_{!*}|_{\mathcal{P}_{!*}(U)} = id$ . Thus  $j^*$  and  $j_{!*}$  are inverse equivalences between  $\mathcal{P}_{!*}(U)$  and  $\mathcal{P}_U$ .  $\square$

From now on assume that  $S = Spec(k)$ , where  $k$  is a field. For a  $G$ -orbit  $O \subset X$  we set  $p(O) = p(x)$ , where  $x$  is a generic point of  $O$  (this number does not depend on the choice of  $x$  because  $x \sim x'$  if  $x, x'$  are generic points of  $O$ ).

**Corollary 4.** For  $\mathcal{F} \in D^b(Coh^G)$  the following statements are equivalent:

- i)  $\mathcal{F}$  is an irreducible object of  $\mathcal{P}^G$ .
- ii) There exists a  $G$ -orbit  $j : O \hookrightarrow X$ , such that  $p(O) < p(x)$ ,  $\bar{p}(O) < \bar{p}(x)$  for any non-generic point  $x \in \bar{O}^{top}$ , and an irreducible  $G$ -equivariant vector bundle  $L$  on  $O$ , such that  $\mathcal{F} = j_{!*}(L[p(O)])$ .

*Proof* (ii)  $\Rightarrow$  (i) is obvious from Lemma 6; let us prove (i)  $\Rightarrow$  (ii). Let  $x$  be a generic point of  $supp(\mathcal{F})$ , and  $Z \subset supp(\mathcal{F})$ ,  $Z \not\ni x$  be a closed ( $G$ -invariant) subscheme. If  $\mathcal{F}$  is irreducible, then for  $\mathcal{G} \in \mathcal{P}_Z$  we have  $Hom(\mathcal{F}, i_{Z*}(\mathcal{G})) = 0$ ,  $Hom(i_{Z*}(\mathcal{G}), \mathcal{F}) = 0$ . Thus Lemma 6 says that  $i_Z^*(\mathcal{F}) \in D^{p_Z, < 0}$ ,  $i_Z^!(\mathcal{F}) \in D^{p_Z, > 0}$ ; and  $i_x^*(\mathcal{F}) \in D^{< p(x)}(\mathcal{O}_x - mod)$ ,  $i_x^!(\mathcal{F}) \in D^{> p(x)}(\mathcal{O}_x - mod)$ . In particular, it follows that  $Z$  can not contain generic points of  $supp(\mathcal{F})$ , hence  $supp(\mathcal{F}) = \bar{G}(x)$ . We also see that if  $x'$  is a non-generic point of  $supp(\mathcal{F})$ , then  $p(x) < p(x')$ ; indeed, otherwise the coherent sheaf  $\mathcal{H}^{p(x)}(\mathcal{F})$  has a nonzero fiber at  $x$ , but has zero fiber at  $x'$ , which contradicts the Nakayama Lemma. Applying the Grothendieck-Serre duality we get also  $\bar{p}(x) < \bar{p}(x')$ . In particular, for any non-generic point  $x' \in (supp(\mathcal{F}))^{top}$  we have  $\dim(x') < \dim(x) - 1 = \dim(supp(\mathcal{F})) - 1$ . Then Rosenlicht's Theorem (see e.g. [VP]) implies that  $x$  is a generic point of an orbit  $O$ . Thus  $\mathcal{F} \in \mathcal{P}_{!*}(O)$ , so (ii) follows from Theorem 2.  $\square$

**Corollary 5.** *Suppose that  $G$  acts on  $X$  with a finite number of orbits, and  $p$  is strictly monotone and comonotone. Then the category  $\mathcal{P}^G$  is Artinian.*

*Proof* Conditions of the Corollary imply that  $j_{!*}^O$  is defined for any  $G$ -orbit  $j^O : O \hookrightarrow X$ . By induction in the number of orbits one can deduce (using Corollary 2) that the irreducible objects  $j_{!*}^O(L[p(O)]) \in \mathcal{P}$  generate the triangulated category  $D^b(\text{Coh})$ . This implies that  $\mathcal{P}$  is Artinian.  $\square$

*Example 1.* Let  $G$  be a simple group over a field of characteristic 0 (or of large finite characteristic), and let  $\mathcal{N} \subset G$  be the subvariety of unipotent elements. Then  $G$  acts on  $\mathcal{N}$  by conjugation, and this action has a finite number of orbits. Moreover, dimension of an orbit is known to be even. Thus the set  $\mathcal{N}^{top}$  consists of generic points of  $G$ -orbits, and we can define the “middle perversity” by  $p(x_O) = -\frac{\dim(O)}{2}$  for an orbit  $O \subset \mathcal{N}$  (where  $x_O$  is the generic point of  $O$ ). Then  $p$  is obviously strictly monotone and comonotone, hence by Proposition 5 the kernel of the corresponding  $t$ -structure is Artinian. See [B] for more information on this example.

*Remark 4.* It will be shown in [AB] that the irreducible objects of the  $t$ -structure described in Example 1 are closely related to cohomology of (tilting) modules over a quantum group at a root of unity. (This relation was independently conjectured by Ostrik).

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